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# A complete conformal metric of preassigned negative Gaussian curvature for a punctured hyperbolic Riemann surface

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**Abstract.** Let $h$ be a complete metric of Gaussian curvature $K_0$ on a punctured Riemann surface of genus $g \geq 1$ (or the sphere with at least three punctures). Given a smooth negative function $K$ with $K = K_0$ in neighbourhoods of the punctures we prove that there exists a metric conformal to $h$ which attains this function as its Gaussian curvature for the punctured Riemann surface. We do so by minimizing an appropriate functional using elementary analysis.

**Keywords.** Punctured Riemann surfaces; prescribed curvature.

## 1. Introduction

Let $\Sigma$ be a punctured hyperbolic Riemann surface, namely a Riemann surface of genus $g \geq 1$ with at least one puncture (or the sphere with at least three punctures). Let $\mathrm{d}s^2 = h\,\mathrm{d}z \otimes \mathrm{d}\bar{z}$ be a complete metric on $\Sigma$ with $K_0 = K_0(z,\bar{z})$ its Gaussian curvature, negative near the punctures. Example of an initial metric is given in §4.

In this paper we prove the result that given an *arbitrary* smooth negative function $K$, with $K = K_0$ in neighbourhoods of the punctures, there exists a metric conformal to $h$ which attains this function as its Gaussian curvature[1]. Our proof is elementary, using Hodge theory, i.e., the existence of the Green's operator for the Laplacian on a compact manifold with boundary. This proof is a generalization of the proof given in an earlier paper that for a compact Riemann surface of genus $g > 1$ any negative function $K$ is attained as the Gaussian curvature of a metric conformal to the given one [10].

In [14] Appendix A, there is a theorem.

**Theorem 1.1 (Hulin–Troyanov).** *Let $S'$ be a non-compact Riemann surface of finite type. Assume $S'$ is neither conformally equivalent to $\mathbb{C}$ nor $\mathbb{C}^*$. Let $K\colon S' \to \mathbb{R}$ be any bounded locally Hölder-continuous function. Then, there exists a conformal metric $g$ on $S'$ with curvature $K$.*

However, such metrics are not usually complete. If we restrict ourselves to complete metrics but forgo having an example in each conformal class, there is a result by Burago (proof is to be found in [14]) generalizing Kazdan and Warner that any function negative

[1] Note that there are minimal surfaces with ends, topologically equivalent to punctured Riemann surfaces, which have negative curvature everywhere [15,22,25].

on an open set with non-positive value somewhere near the punctures is achieved as a prescribed Gaussian curvature [5,14,18].

**Theorem 1.2.(Burago–Kazhdan–Warner).** *Let $K$ be a smooth function on an open surface $S'$ of finite type. The following conditions are necessary and sufficient for $K$ to be the curvature of some complete metric of $S'$:*

(i) $\liminf_{x\to\infty} K(x) \le 0$ *at each end of* $S'$;
(ii) *when Euler characteristic of* $S' < 0$, *assume also,* $\inf K < 0$; *when Euler characteristic of* $S' = 0$, *assume also* $\inf K < 0$ *or* $K \equiv 0$.

The proof of this theorem is also given in [14]. However they do not find a metric in every conformal class which has this property.

Our proof is very different and our result is new in the sense that we show that if the initial metric has negative Gaussian curvature near the punctures and given any negative function with the same value as the initial one near the punctures, there exists a metric in the conformal class of the initial metric which has this function as its Gaussian curvature. Moreover, the behaviour of the initial metric near the punctures do not change (the conformal equivalence factor $\mathrm{e}^{\sigma}$ is 1 near the punctures). Thus if we started with a complete metric, completeness is preserved. If the *initial* metric has Gaussian curvature the function $K_0$, negative somewhere, such that $\lim_{x\to\infty} K_0 = f_0 < 0$ (i.e. its value is $f_0$ near the punctures) then by Burago–Kazhdan–Warner's result we would have a complete metric which achieves $K_0$ as its Gaussian curvature. Let $K < 0$ be another function such that $\lim_{x\to\infty} K = f_0$ then our result shows that there exists a metric conformal to the previous one such that $K$ is achieved as its Gaussian curvature. This indicates that there could be many conformal classes which contain a metric with $K$ as its Gaussian curvature. In the last section we give an example of the initial metric which has negative Gaussian curvature near the punctures.

By Thereom 2.1a, we also have the result that any arbitrary negative function $K$ is allowed as the Gaussian curvature of an unpunctured Riemann surface of genus $g \ge 1$ with at least one boundary component or a sphere with at least three boundary components.

For prescribed curvature on surfaces with conical singularities, see [23].

Let $p_i$, $i = 1,\dots,n$ be the punctures on $\Sigma$. Let us choose a disc $D_i$ about the point $p_i$ with coordinates $(r_i,\theta_i)$ such that $D_i = \{(r_i,\theta_i) : 0 < r_i < 1,\ 0 < \theta_i \le 2\pi\}$ with the point $p_i$ corresponds to $r_i = 0$. No other puncture lies in this neighbourhood. Let $M = \Sigma - \cup_{i=1}^{n} D_i$. Thus $M$ is a compact manifold with boundary $\partial M = \cup_{i=1}^{n} \partial D_i$. We assume $K_0$ is negative on $\partial M$. In §4 we show by an example that this is not a stringent condition on the metric $h$. Also $M$ is of finite volume with respect to the metric $h$ and $K_0$ is smooth on $M$. $K_0$ may have singularities at the punctures. Let $K < 0$ be a given function on $M$ and $K = K_0$ on $\cup_{i=1}^{n} D_i$. On $M$ we show that there exists a metric conformal to $h$ which has $K$ as its Gaussian curvature. We show this by minimizing the functional

$$S[\sigma] = \int_M (K(\sigma) - K)^2 \mathrm{e}^{2\sigma} \mathrm{d}\mu$$

over functions in class $T$ where $T = \{\sigma \in C^2(M,\mathbb{R}) | \sigma|_{\partial M} = \partial_\nu \sigma|_{\partial M} = \Delta\sigma|_{\partial M} = 0\}^-$ where $-$ denotes the closure in $W^{2,2}$. Here $K(\sigma)$ stands for the Gaussian curvature of the metric $\mathrm{e}^{\sigma} h$, and $\mathrm{d}\mu = \frac{\sqrt{-1}}{2} h\ \mathrm{d}z \wedge \mathrm{d}\bar{z}$ is the area form for the metric $h$. Using the Sobolev embedding theorem for compact manifolds with boundary we show that $S[\sigma]$ takes its

absolute minimum, zero, on $C^\infty(M)$ which corresponds to a metric on $M$ of negative curvature $K$. Then we extend the minimizer $\sigma$ by zero on $D_i$ and thus near the punctures the metric $\mathrm{e}^\sigma h\,\mathrm{d}z \otimes \mathrm{d}\overline{z}$ and the curvature $K$ remain $h\,\mathrm{d}z \otimes \mathrm{d}\overline{z}$ and $K_0$ respectively.

## 2. The main theorem

2.1 *All notations are as in* §1

The functional $S[\sigma] = \int_M (K(\sigma) - K)^2 \mathrm{e}^{2\sigma} \mathrm{d}\mu$ is non-negative on $T$, so that its infimum

$$S_0 = \inf\{S[\sigma], \sigma \in T\}$$

exists and is non-negative. Let $\{\sigma_n\}_{n=1}^\infty \subset T$ be a corresponding minimizing sequence,

$$\lim_{n\to\infty} S[\sigma_n] = S_0.$$

Our main result is the following

**Theorem 2.1.** (a) *Let $M = \Sigma - \cup_{i=1}^n D_i$ be a compact Riemann surface with boundary, with at least three boundary components if genus $g = 0$. Let $h\,\mathrm{d}z \wedge \mathrm{d}\overline{z}$ be its initial metric and $K_0$ be its initial curvature such that $K_0 < 0$ on $\cup_{i=1}^n D_i$. Let $K < 0$ be an arbitrary negative function with $K = K_0$ on $\cup_{i=1}^n D_i$ (i.e., the curvature is left intact near the punctures). The infimum $S_0 = 0$ is attained at a unique $\sigma \in C^\infty(M,\mathbb{R})$, i.e., the minimizing sequence $\{\sigma_n\}$ contains a subsequence that converges in strong $W^{2,2}$ to a unique $\sigma \in C^\infty(M,\mathbb{R})$ such that $S[\sigma] = S_0 = 0$. The corresponding metric $\mathrm{e}^\sigma h\,\mathrm{d}z \otimes \mathrm{d}\overline{z}$ is a metric on $M$ of preassigned negative curvature $K$.* (b) *On $\cup_{i=1}^n D_i$, $\sigma$ is extended as zero.*

2.2 *Uniform bounds*

Since $\{\sigma_n\}$ is a minimizing sequence, we have the obvious inequality

$$S[\sigma_n] = \int_M (K_n - K)^2 \mathrm{e}^{2\sigma_n} \mathrm{d}\mu = \int_M \left(K_0 - \frac{1}{2}\Delta_h \sigma_n - K\mathrm{e}^{\sigma_n}\right)^2 \mathrm{d}\mu \le m \qquad (2.1)$$

for some $m > 0$, where we denoted by $K_n$ the Gaussian curvature $K(\sigma_n)$ of the metric $\mathrm{e}^{\sigma_n} h$ and by $K_0$ that of the metric $h$, and used that

$$K_n = \mathrm{e}^{-\sigma_n}\left(K_0 - \frac{1}{2}\Delta_h \sigma_n\right).$$

*Note:* Here $\Delta_h = 4h^{-1}(\partial^2/\partial z \partial \overline{z})$ stands for the Laplacian defined by the metric $h$ on $M$.

*Lemma* 2.2. *There exists a constant $C_1$ such that, uniformly in $n$,*

$$\int_M (\Delta_h \sigma_n)^2 \mathrm{d}\mu < C_1.$$

*Proof.* By Minkowski inequality, and using (2.1), we get

$$\left[\int_M \left(-\frac{1}{2}\Delta_h\sigma_n - K\mathrm{e}^{\sigma_n}\right)^2 \mathrm{d}\mu\right]^{1/2} \leq \left[\int_M \left(K_0 - \frac{1}{2}\Delta_h\sigma_n - K\mathrm{e}^{\sigma_n}\right)^2 \mathrm{d}\mu\right]^{1/2} + \left[\int_M (K_0)^2 \mathrm{d}\mu\right]^{1/2} \leq m^{1/2} + c = C,$$

so that

$$\frac{1}{4}\int_M (\Delta_h\sigma_n)^2\mathrm{d}\mu + \int_M K^2\mathrm{e}^{2\sigma_n}\mathrm{d}\mu + \int_M \Delta_h\sigma_n\mathrm{e}^{\sigma_n}K\mathrm{d}\mu \leq C^2. \tag{2.2}$$

We will show that

$$\int_M K^2\mathrm{e}^{2\sigma_n}\mathrm{d}\mu + \int_M \Delta_h\sigma_n\mathrm{e}^{\sigma_n}K\mathrm{d}\mu = B_{n1} + B_{n2} + B_{n3}, \tag{2.3}$$

where $B_{n1} \geq 0$, $B_{n2} \geq 0$, $|B_{n3}| \leq 3D^2$ where $D^2$ is a constant independent of $n$.

From (2.3) and (2.2) the result will follow since we will have

$$\begin{aligned} C^2 + 3D^2 \geq C^2 - B_{n3} &\geq \frac{1}{4}\int_M (\Delta_h\sigma_n)^2\mathrm{d}\mu + B_{n1} + B_{n2} \\ &\geq \frac{1}{4}\int_M (\Delta_h\sigma_n)^2\mathrm{d}\mu. \end{aligned}$$

Just renaming the constants, we will have the result.

Integrating by parts we get

$$\begin{aligned} \int_M \Delta_h\sigma_n\mathrm{e}^{\sigma_n}K\mathrm{d}\mu &= -\int_M |\partial_z\sigma_n|^2\mathrm{e}^{\sigma_n}K\mathrm{d}\mu - \int_M (\partial_z\sigma_n)(\partial_{\bar{z}}K)\mathrm{e}^{\sigma_n}\mathrm{d}\mu \\ &\quad - \int_{\partial M} (\partial_\nu\sigma_n)K\mathrm{e}^{\sigma_n}\mathrm{d}\mu \\ &= \int_M |\partial_z\sigma_n|^2\mathrm{e}^{\sigma_n}|K|\mathrm{d}\mu - \int_M (\partial_z\sigma_n)g|K|\ \mathrm{e}^{\sigma_n}\mathrm{d}\mu \end{aligned}$$

since $K$ is negative, $\partial_\nu\sigma_n|_{\partial M} = 0$ and where we define $g = \partial_{\bar{z}}K/|K|$.

Let $M = \Omega_{n1} \cup \Omega_{n2} \cup \Omega_{n3}$, a disjoint union of sets defined as follows:

On $\Omega_{n1}$, (1) $|\partial_z\sigma_n| > |g|$.

On $\Omega_{n2}$, (2) $|\partial_z\sigma_n| \leq |g|$ and $|K|\ \mathrm{e}^{\sigma_n} > |g|^2$.

On $\Omega_{n3}$, (3) $|\partial_z\sigma_n| \leq |g|$ and $|K|\ \mathrm{e}^{\sigma_n} \leq |g|^2$.

Let $B_{ni} = \int_{\Omega_{ni}} K^2\mathrm{e}^{2\sigma_n}\mathrm{d}\mu + \int_{\Omega_{ni}} \Delta_h\sigma_n\mathrm{e}^{\sigma_n}K\mathrm{d}\mu,\ i = 1,2,3.$

We will show that $B_{n1} \geq 0$.

$$\begin{aligned} B_{n1} &= \int_{\Omega_{n1}} K^2\mathrm{e}^{2\sigma_n}\mathrm{d}\mu + \int_{\Omega_{n1}} |\partial_z\sigma_n|^2|K|\ \mathrm{e}^{\sigma_n}\mathrm{d}\mu - \int_{\Omega_{n1}} (\partial_z\sigma_n)g|K|\ \mathrm{e}^{\sigma_n}\mathrm{d}\mu \\ &\geq \int_{\Omega_{n1}} K^2\mathrm{e}^{2\sigma_n}\mathrm{d}\mu + \int_{\Omega_{n1}} |\partial_z\sigma_n|^2|K|\ \mathrm{e}^{\sigma_n}\mathrm{d}\mu - \int_{\Omega_{n1}} |\partial_z\sigma_n||g||K|\ \mathrm{e}^{\sigma_n}\mathrm{d}\mu \end{aligned}$$

$$= \int_{\Omega_{n1}} K^2 e^{2\sigma_n} d\mu + \int_{\Omega_{n1}} |\partial_z \sigma_n||K| \, e^{\sigma_n} (|\partial_z \sigma_n| - |g|) d\mu$$
$$\geq 0$$

by (1) in the definition of $\Omega_{n1}$.

Next we shall show that $B_{n2} \geq 0$.

$$\begin{aligned}
B_{n2} &= \int_{\Omega_{n2}} K^2 e^{2\sigma_n} d\mu + \int_{\Omega_{n2}} |\partial_z \sigma_n|^2 |K| \, e^{\sigma_n} d\mu - \int_{\Omega_{n2}} (\partial_z \sigma_n) g |K| \, e^{\sigma_n} d\mu \\
&\geq \int_{\Omega_{n2}} K^2 e^{2\sigma_n} d\mu + \int_{\Omega_{n2}} |\partial_z \sigma_n|^2 |K| \, e^{\sigma_n} d\mu - \int_{\Omega_{n2}} |\partial_z \sigma_n||g||K| \, e^{\sigma_n} d\mu \\
&= \int_{\Omega_{n2}} |K| \, e^{\sigma_n} (|K| \, e^{\sigma_n} - |\partial_z \sigma_n||g|) d\mu + \int_{\Omega_{n2}} |\partial_z \sigma_n|^2 e^{\sigma_n} |K| d\mu \\
&\geq \int_{\Omega_{n2}} |K| \, e^{\sigma_n} (|K| \, e^{\sigma_n} - |g|^2) d\mu + \int_{\Omega_{n2}} |\partial_z \sigma_n|^2 e^{\sigma_n} |K| d\mu \\
&\geq 0,
\end{aligned}$$

by using the two conditions (2) defining $\Omega_{n2}$.

Next we shall show that $B_{n3}$ is uniformly bounded.

$$\begin{aligned}
|B_{n3}| &\leq \int_{\Omega_{n3}} K^2 e^{2\sigma_n} d\mu + \int_{\Omega_{n3}} |\partial_z \sigma_n|^2 |K| \, e^{\sigma_n} d\mu + \int_{\Omega_{n3}} |\partial_z \sigma_n||g||K| \, e^{\sigma_n} d\mu \\
&\leq 3D^2,
\end{aligned}$$

where $D^2 = \max |g|^4 \mu(M)$, where $\mu(M)$ is the volume of $M$. This follows from the two conditions $|K| \, e^{\sigma_n} \leq |g|^2$ and $|\partial_z \sigma_n| \leq |g|$ on $\Omega_{n3}$. $D^2$ is a finite constant ($\max |g|^4$ is finite since $K$ is non-zero and the volume of $M$ is finite), independent of $n$. Thus the result follows.

## 2.3 *Pointwise convergence of* $\sigma_n$

*Lemma* 2.3. *$\{\sigma_n\}$ is uniformly bounded in $W^{2,2}(M)$.*

*Proof.* Let us recall that there is a Green's function on the compact manifold $M$ with boundary such that if $u \in C^2(M)$ then

$$-u(x) = \int \int_M G(x,z) \Delta u(z) d\mu(z) + \int_{\partial M} \frac{\partial G}{\partial \nu}(x,w) u(w) dl$$

(see for e.g. [9], p. 174). In particular if $u|_{\partial M} = 0$ then

$$-u(x) = \int \int_M G(x,z) \Delta u(z) d\mu(z).$$

Thus there exists a Green's operator $G$ such that $\Delta G u = -u$ and $G \Delta u = -u$ (see [9], p. 177). Let $u = \sigma_n$. Since $\sigma_n|_{\partial M} = 0$, $G \Delta \sigma_n = -\sigma_n$.

Next we will show that since by Lemma 2.2, $\Delta \sigma_n$ is uniformly bounded in $L^2(M)$, $\sigma_n$ is uniformly bounded in $W^{2,2}(M)$.

Let $\tau_n = \Delta\sigma_n$. We know that $\tau_n|_{\partial M} = 0$ and is uniformly bounded in $L^2(M)$, i.e., $\|\tau_n\|_2 < C, \forall n$. Then $\tau_n$ can be written as $\tau_n = \sum_{i=1}^{\infty} c_n^i \phi_i$ where $\phi_i$ is a solution to the equation $\Delta_h \phi + \lambda_i \phi = 0$, with $\phi|_{\partial M} = 0$, ([9], pp. 8–9). $\{\phi_i\}_{i=1}^{\infty}$ is an orthonormal sequence of eigenfunctions of the Laplacian. In fact since we are in the Dirichlet case, the least eigenvalue $\lambda_1 > 0$. We will show that since $\{\tau_n\}$ is uniformly bounded in $L^2(M)$, $\{G\tau_n\}$ is uniformly bounded in $W^{2,2}(M)$. $G\tau_n = \sum_{i=1}^{\infty} c_n^i G\phi_i$. Since $\phi_i|_{\partial M} = 0$, $G\Delta\phi_i = -\phi_i$. Since $\Delta_h \phi_i = -\lambda_i \phi_i$, $G\phi_i = \phi_i/\lambda_i$. Thus $G\tau_n = \sum_{i=1}^{\infty} (c_n^i/\lambda_i)\phi_i$. Since $\lambda_i \to \infty$ and since $\|\tau_n\|_2 = (\sum_{i=1}^{\infty} c_n^{i2})^{1/2} < C$, $\|G\tau_n\|_2 = (\sum_{i=1}^{\infty} (c_n^i/\lambda_i)^2)^{1/2} < C$, $\forall n$ if $\lambda_1 > 1$. If $\lambda_1 < 1$, $\|G\tau_n\|_2^2 \le (C^2/\lambda_1^2)$ $\forall n$. Thus $\|G\tau_n\| \le C_2$ $\forall n$ where

$$
\begin{aligned}
C_2 &= C \quad \text{if } \lambda_1 > 1 \\
&= \frac{C}{\lambda_1} \quad \text{if } \lambda_1 < 1.
\end{aligned}
$$

Next we show that the first and second derivative norms of $G\tau_n$ are uniformly bounded in $L^2(M)$. The second derivative is defined as $\nabla^2 \tau = \nabla^1 \nabla \tau$ where $\nabla \tau$ is the usual covariant derivative of $f$ taking values in $T^*M$ and $\nabla^1$ is the covariant derivative of sections of $T^*M$.

First we show that the first derivative norm is uniformly bounded. For this we observe that the norm of the Laplacian is bounded uniformly. This is because $\Delta_h G\tau_n = -\tau_n$ and $\tau_n$ is uniformly bounded in $L^2(M)$. Thus $\|\Delta_h G\tau_n\|_2 < C$.

To show that $\|\nabla G\tau_n\|_2^2 < C^2$ we note that since $G\tau_n$ is zero on $\partial M$,

$$
\begin{aligned}
\int_M |\nabla G\tau_n|^2 \mathrm{d}\mu &= -\int_M G\tau_n \Delta_h G\tau_n \\
&\le \|G\tau_n\|_2 \|\Delta G\tau_n\|_2 \\
&\le C_2 C.
\end{aligned}
$$

Next we show that the second derivative norm $\|\nabla^2 G\tau_n\|_2 < C_1$, $\forall n$. The norm of the Laplacian is bounded in $L^2(M)$. Using Weitzenbock formula we show that then the second derivative norm is uniformly bounded. By Weitzenbock formula [4], $\Delta_h = \nabla^* \nabla + \tau(K_0)$ where $\nabla^*$ is the adjoint of the covariant derivative. Note that $\nabla^*\nabla = -\nabla^1\nabla = -\nabla^2$ ([4], p. 52). $\tau(K_0)$ is some continuous function of the curvature $K_0$. In other words, $\Delta_h(G\tau_n) = -\nabla^2(G\tau_n) + \tau(K_0)(G\tau_n)$. Thus by Minkowski's inequality,

$$
\begin{aligned}
\|\nabla^2(G\tau_n)\|_2 &\le \|\Delta_h(G\tau_n)\|_2 + \|\tau(K_0)G\tau_n\|_2 \\
&\le C + \min_M |\tau(K_0)| \|G\tau_n\|_2 \\
&\le C + \min_M |\tau(K_0)| C_2 \\
&\le C_1.
\end{aligned}
$$

Since $M$ is a compact manifold with boundary, $\min_M |\tau(K_0)|$ is bounded.

Thus $G\tau_n$ is uniformly bounded in $W^{2,2}$ since $\|G\tau_n\|_2$, $\|\nabla(G\tau_n)\|_2$, $\|\nabla^2(G\tau_n)\|_2$ are uniformly bounded. Since $\tau_n = \Delta_h \sigma_n$ and $G\Delta_h \sigma_n = \sigma_n$, $\sigma_n$ are uniformly bounded in $W^{2,2}$.

Now we can formulate the main result of this subsection.

PROPOSITION 2.4.

*The sequence* $\{\sigma_n\}_{n=1}^{\infty}$ *contains a subsequence* $\{\sigma_{l_n}\}_{n=1}^{\infty}$ *with the following properties*:

(a) *The sequences* $\{\sigma_{l_n}\}_{n=1}^{\infty}$ *and* $\{\mathrm{e}^{\sigma_{l_n}}\}$ *converge in* $C^0(M)$ *topology to continuous functions* $\sigma$ *and* $\mathrm{e}^{\sigma}$ *respectively. Moreover,* $\sigma \in W^{2,2}(M)$.
(b) *The subsequence* $\{\Delta_h \sigma_{l_n}\}$ *converges weakly in* $L^2$ *to* $f \doteq \Delta_h^{\mathrm{distr}} \tilde{\sigma}$ *– a distribution Laplacian of* $\sigma$.
(c) *Passing to this subsequence* $\{\sigma_{l_n}\}$, *the following limits exist*:

$$\lim_{n\to\infty} \|\Delta_h \sigma_{l_n}\|_2 = \|\Delta_h^{\mathrm{distr}} \sigma\|_2,$$

$$\lim_{n\to\infty} S[\sigma_{l_n}] = S_0 = \int_M \left(K_0 - \frac{1}{2}\Delta_h^{\mathrm{distr}}\sigma - K\mathrm{e}^{\sigma}\right)^2 \mathrm{d}\mu.$$

*In fact, the convergence in* (b) *is strong in* $L^2$.

*Proof.* Part (a) follows from the Sobolev embedding theorem and Rellich lemma for compact manifolds with boundary, since, for $\dim M = 2$, the space $W^{2,2}(M)$ is compactly embedded into $C^0(M)$ (see, e.g. [2]). Therefore the sequence $\{\sigma_n\}$, which according to Lemma 2.3, is uniformly bounded in $W^{2,2}(M)$, contains a convergent subsequence in $C^0(M)$. Passing to this subsequence $\{\sigma_{l_n}\}$ we can assume that there exists a function $\sigma \in C^0(M)$ such that

$$\lim_{n\to\infty} \sigma_{l_n} = \sigma.$$

Since $\sigma_n$'s are uniformly bounded in a Hilbert space $W^{2,2}(M)$, they weakly converge to $s \in W^{2,2}(M)$ (after passing to a subsequence if necessary). The uniform limit coincides with $s$ so that $\sigma = s \in W^{2,2}(M)$.

In order to prove (b), set $\psi_n = \Delta_h \sigma_{l_n}$ and observe that, according to part (a) of Lemma 2.2, the sequence $\{\psi_n\}$ is bounded in $L^2$. Therefore, passing to a subsequence, if necessary, there exists $f \in L^2(M)$ such that

$$\lim_{n\to\infty} \int_M \psi_n g = \int_M fg$$

for all $g \in L^2(M)$. In particular, considering $g \in C^\infty(M)$, this implies $f = \Delta_h^{\mathrm{distr}}\sigma$.

In order to prove (c) we use the following lemma.

*Lemma* 2.5. *If a sequence* $\{\psi_n\}$ *converges to* $f \in L^2$ *in the weak topology, then*

$$\lim_{n\to\infty} \|\psi_n\| \geq \|f\|.$$

*Further* $\lim_{n\to\infty} \|\psi_n\| = \|f\|$ *iff there is strong convergence.*

*Proof.* The lemma follows from considering the following inequality:

$$\lim_{n\to\infty} \int (\psi_n - f)^2 \mathrm{d}\mu \geq 0.$$

To continue with the proof of the proposition, suppose $\lim_{n\to\infty}\|\psi_n\| > \|f\|$. Using the definition of the functional, we have

$$S[\sigma_n] = \int_M \left(K_0 - \frac{1}{2}\Delta_h\sigma_n - K\mathrm{e}^{\sigma_n}\right)^2 \mathrm{d}\mu$$
$$= \frac{1}{4}\|\psi_n\|^2 + \|K_0 - K\mathrm{e}^{\sigma_n}\|^2 - \int_M \psi_n(K_0 - K\mathrm{e}^{\sigma_n})\mathrm{d}\mu.$$

From parts (a) and (b) it follows that the sequence $S[\sigma_n]$ converges to $S_0$ and

$$S_0 = \lim_{n\to\infty} S[\sigma_n]$$
$$= \lim_{n\to\infty}\frac{1}{4}\|\psi_n\|^2 + \|K_0 - K\mathrm{e}^{\sigma}\|^2 - \int_M f(K_0 - K\mathrm{e}^{\sigma})\mathrm{d}\mu$$
$$\geq \frac{1}{4}\|f\|^2 + \|K_0 - K\mathrm{e}^{\sigma}\|^2 - \int_M f(K_0 - K\mathrm{e}^{\sigma})\mathrm{d}\mu$$
$$= \left\| -\frac{1}{2}f + K_0 - K\mathrm{e}^{\sigma}\right\|^2.$$

We will show that this is an equality since the inequality contradicts the fact that $\{\sigma_n\}$ was a minimizing sequence. That is we can construct a sequence $\{\tau\} \in C^\infty(M)$ such that $S[\tau]$ gets as close to $\left\| -\frac{1}{2}f + K_0 - K\mathrm{e}^{\sigma}\right\|^2$ as we like.

Namely, for any $\varepsilon > 0$ we can construct, by the density of $C^\infty$ in $W^{2,2}$, a function $\tau \in C^\infty(M)$ approximating $\sigma \in W^{2,2}$ such that $\|\Delta_h\tau - f\| < \varepsilon$ and $\|4(\mathrm{e}^\tau - \mathrm{e}^\sigma)\| < \varepsilon/2$. Since

$$S_\tau = \lim_{n\to\infty} S[\tau] = \left\| -\frac{1}{2}\Delta_h\tau + K_0 - K\mathrm{e}^{\tau}\right\|^2,$$

we have

$$\left\| \sqrt{S_\tau} - \left\| -\frac{1}{2}f + K_0 - K\mathrm{e}^{\sigma}\right\| \right\| \leq \left\| \frac{1}{2}(f - \Delta_h\tau) - K(\mathrm{e}^\tau - \mathrm{e}^\sigma)\right\| \leq \varepsilon.$$

Now setting $\delta = \sqrt{S_0} - \| -\frac{1}{2}f + K_0 - K\mathrm{e}^{\sigma}\| > 0$ and choosing $\varepsilon < \delta/2$, and using $\sqrt{S_\tau} \leq \| -\frac{1}{2}f + K_0 - K\mathrm{e}^{\sigma}\| + \varepsilon$ we get, $\sqrt{S_\tau} < \sqrt{S_0} - \frac{\delta}{2}$ – a contradiction, since $S_0$ is the infimum of the functional.

Thus, $\lim_{n\to\infty}\|\Delta_h\sigma_n\| = \|f\|$, so that, in fact, by Lemma 2.5, the convergence is in the strong $L^2$ topology. This proves part (c).

## 3. Smoothness and uniqueness

Here we complete the proof of the main Theorem 2.1 by showing that

PROPOSITION 3.1.

(a) *The minimizing function $\sigma \in C^0(M)$ is smooth and unique and corresponds to a metric of negative curvature $K$.*
(b) *$\sigma$ extends to $M - \cup_{i=1}^n p_i$ by zero.*

*Proof.* (a) Let $b = (K_0 - \frac{1}{2}\Delta_h^{\text{distr}}\sigma - K\mathrm{e}^{\sigma}) \in L^2(M)$; note that $b = 0$ on the boundary $\partial M$, since $\sigma|_{\partial M} = 0$ and by redefining $\Delta_h^{\text{distr}}\sigma$ on the boundary, we can take $\Delta_h^{\text{distr}}\sigma|_{\partial M} = 0$ and we know that $K|_{\partial M} = K_0$. According to Proposition 2.4, (c) $S_0 = \int_M b^2 \mathrm{d}\mu$. Set $G(t) = S(\sigma + t\beta) - S_0$, where $\beta \in \{f \in C^\infty(M) | \partial_\nu f|_{\partial M} = 0, f|_{\partial M} = 0\}$. $G(t)$ for fixed $\beta$ is smooth, $G(0) = 0$ and $G(t) \geq 0$ for all $t$. Therefore,

$$\left.\frac{\mathrm{d}G}{\mathrm{d}t}\right|_{t=0} = 0.$$

A simple calculation yields

$$\left.\frac{\mathrm{d}G}{\mathrm{d}t}\right|_{t=0} = \int_M (-b\Delta_h\beta - 2K\mathrm{e}^{\sigma} b\beta)\mathrm{d}\mu.$$

Now, $\int_M b\Delta_h\beta \ \mathrm{d}\mu - \int_M \beta\Delta_h b \ \mathrm{d}\mu = \int_{\partial M} b\partial_\nu\beta \ \mathrm{d}l - \int_{\partial M} \beta\partial_\nu b \ \mathrm{d}l = 0$. Thus $b \in L^2(M)$ satisfies, in a distributional sense, the following equation:

$$-\Delta_h b - 2K\mathrm{e}^{\sigma} b = 0. \tag{3.1}$$

First, we will show that $b = 0$ is the only weak $L^2$ solution to eq. (3.1). Indeed, by elliptic regularity $b$ is smooth, so that multiplying (3.1) by $b$ and integrating over $M$ using the Stokes formula, we get

$$\int_M \nabla b \cdot \nabla b - \int_{\partial M} \partial_\nu b \cdot b - \int_M 2Kb^2\mathrm{e}^{\sigma}\mathrm{d}\mu = 0.$$

The second term drops since we redefined $\Delta^{\text{distr}}\sigma$ to be zero on the boundary and had $K = K_0$ on the boundary, thus $b = 0$ on $\partial M$. Recalling that $K < 0$ on $M$, all the remaining terms are positive or zero, $b = 0$ everywhere. Thus we have shown that $S_0 = 0$.

Secondly, equation $b = 0$ for the minimizing function $\sigma \in C^0(M)$ reads

$$\frac{1}{2}\Delta_h^{\text{distr}}\sigma = K_0 - K\mathrm{e}^{\sigma} \in C^0(M). \tag{3.2}$$

Therefore, $\Delta_h^{\text{distr}}\sigma$ belongs to $L^p(M)$ so that $\sigma \in W^{2,p}$ for all $p$. By the Sobolev embedding theorem it follows that $\sigma \in C^{1,\alpha}(M)$ for some $0 < \alpha < 1$. Therefore, the right-hand side of eq. (3.3) actually belongs to the space $C^{1,\alpha}(M)$, and therefore $\sigma \in C^{3,\alpha}(M)$ and so on. This kind of bootstrapping argument shows that $\sigma$ is smooth.

Equation $b \equiv 0$ satisfied by $\sigma$ now translates to $K(\sigma) \equiv K$, where $K(\sigma)$ is the Gaussian curvature of the metric $\mathrm{e}^{\sigma} h \ \mathrm{d}z \otimes \mathrm{d}\bar{z}$, $\sigma \in C^\infty(M)$.

Next to show uniqueness, let $\eta$ be another solution in class $T$. Then $\eta$ satisfies

$$\frac{1}{2}\Delta_h^{\text{distr}}\eta = K_0 - K\mathrm{e}^{\eta} \in C^0(M) \tag{3.3}$$

so that

$$\Delta_h(\sigma - \eta) = -2K(\mathrm{e}^{\sigma} - \mathrm{e}^{\eta}).$$

Multiplying this equation by $\sigma - \eta$ and remembering that $\sigma$ and $\eta$ and their normal derivatives vanish on the boundary, we get

$$-\int_M \mathrm{d}\zeta \wedge *\mathrm{d}\zeta = \int_M -2K(\sigma - \eta)(\mathrm{e}^{\sigma} - \mathrm{e}^{\eta})\mathrm{d}\mu,$$

where we set $\zeta = \sigma - \eta$. Since $-2K(\sigma - \eta)(\mathrm{e}^{\sigma} - \mathrm{e}^{\eta}) \geq 0$, we conclude that $\mathrm{d}\zeta = 0$ and in fact $\zeta = 0$.

(b) $\sigma$ is in class $T$, i.e. $\sigma|_{\partial M} = \partial_\nu \sigma|_{\partial M} = \Delta_h \sigma|_{\partial M} = 0$. Supposing we minimize over the class of functions all of whose derivatives are zero at $\partial M$. Let $\tau$ be the smooth function to which this minimizing sequence converge to. By uniqueness of the solution, $\tau = \sigma$. We can extend $\tau$ by zero on the discs $\cup_{i=1}^n D_i$. Thus we can extend $\sigma$ by zero as well.

## 4. An example of the initial metric

Let $\Sigma - \{p_i\}_{i=1}^n$ be the Riemann surface with punctures. Let $D_i = \{(r_i, \theta_i) : 0 < r_i < 1\}$ be a disc about $p_i$ such that $p_i$ is the only puncture on $D_i$. Recall $M = \Sigma - \cup_{i=1}^n D_i$. $\partial M = \cup_{i=1}^n \partial D_i = \cup_{i=1}^n \{(r_i, \theta_i) : r_i = 1\}$. Let $D_i' = \{(r_i, \theta_i) : 0 < r_i < 1.5\}$ and $D_i'' = \{(r_i, \theta_i) : 0 < r_i < 2\}$. Let $f_i \; \mathrm{d}z \otimes \mathrm{d}\overline{z}$ be a metric on $D_i'' - \; p_i$. Let $\tilde{h} \; \mathrm{d}z \otimes \mathrm{d}\overline{z}$ be a metric on $M$. We wish to construct a metric which will interpolate between the metric on $M$ and the metric on $D_i''$, $i = 1, \ldots, n$. Let

$$
\begin{aligned}
\rho_i &= 1 \quad \text{on } \{0 \leq r_i \leq 1.5\} \\
&= 0 \quad \text{on } \{r_i \geq 2\} \\
&\quad 0 < \rho < 1 \quad \text{on } \{1.5 < r_i < 2\}
\end{aligned}
$$

and

$$
\begin{aligned}
\tilde{\rho} &= 1 \quad \text{on } \Sigma - \cup_{i=1}^n D_i'' \\
&= 0 \quad \text{on } D_i' \; \forall i \\
&\quad 0 < \tilde{\rho} < 1 \quad \text{on } \{1.5 < r_i < 2\}.
\end{aligned}
$$

Let

$$
h \; \mathrm{d}z \otimes \mathrm{d}\overline{z} = \sum_{i=1}^n \rho_i f_i \; \mathrm{d}z \otimes \mathrm{d}\overline{z} + \tilde{\rho} \tilde{h} \; \mathrm{d}z \otimes \mathrm{d}\overline{z}.
$$

This is a metric because $h$ is positive everywhere. On $D_i'$, $h \; \mathrm{d}z \otimes \mathrm{d}\overline{z} = f_i \; \mathrm{d}z \otimes \mathrm{d}\overline{z}$ and on $\Sigma - \cup_{i=1}^n D_i''$, $h \; \mathrm{d}z \otimes \mathrm{d}\overline{z} = \tilde{h} \; \mathrm{d}z \otimes \mathrm{d}\overline{z}$.

We can choose $f_i$ such that the Gaussian curvature $K_0$ of $h \; \mathrm{d}z \otimes \mathrm{d}\overline{z}$ is negative on $\partial M$. One such example would be to choose $f_i \; \mathrm{d}z \otimes \mathrm{d}\overline{z}$ on the disc $D_i' - p_i$ to be $-\log(\frac{r_i}{4})[(\mathrm{d}r_i)^2 + r_i^2 (\mathrm{d}\theta_i)^2]$, where $(r_i, \theta_i)$ are standard polar coordinates on the disc. This metric has a true singularity at the puncture. $K_0 = \frac{1}{2\sqrt{EG}} \left[ \left( \frac{E_{\theta_i}}{\sqrt{EG}} \right)_{\theta_i} + \left( \frac{G_{r_i}}{\sqrt{EG}} \right)_{r_i} \right]$ where $E = -\log\left(\frac{r_i}{4}\right)$, $G = -r_i^2 \log\left(\frac{r_i}{4}\right)$ [11]. On $D_i$, $K_0 = \frac{1}{2r_i^2 (\log(r_i/4))^3}$ and therefore negative. In particular, on $\partial M = \cup_{i=1}^n \{(r_i, \theta_i) : r_i = 1\}$, $K_0$ is negative.

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